Solving Nonhomogeneous equations I

Undetermined coefficients

This technique grows out of the "trial-and-error method"

Find a particular solution of $y''+y'+2y = \sin(x)$

$p(D) = D^2+D+2$ Want to solve $p(D)y = \sin(x)$

Why not try a multiple of $\sin(x)$, $A\sin(x)$?

$$\begin{aligned} p(D)(A\sin(x)) &= A\sin''(x) + A\sin'(x) + 2A\sin(x) \\ &= -A\sin(x) + A\cos(x) + 2A\sin(x) \\ &= A\sin(x) + A\cos(x) \end{aligned}$$

If we take $A=1$, we get $\sin(x)$, but also $\cos(x)$.
So why not include a $\cos(x)$ term as well?

$$p(D)(A\sin(x)+B\cos(x)) = (D^2+D+2)(A\sin(x)+B\cos(x))$$

$$= -A\sin(x) - B\cos(x) + A\cos(x) - B\sin(x) + 2A\sin(x) + 2B\cos(x)$$

$$= (-A-B+2A)\sin(x) + (-B+A+2B)\cos(x)$$

$$= (A-B)\sin(x) + (A+B)\cos(x)$$

Want this $= \sin(x)$, so want $\begin{array}{l} A-B=1 \\ A+B=0 \end{array}$ $\quad \begin{array}{l} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{array}$