

Example: $y'' - 4y = \sin(x)$

(a) Find general solution.

(b) Solve initial value problem $y(0) = 0$
 $y'(0) = 0$

a Particular solution is known $y_p(x) = -\frac{1}{5} \sin(x)$

The general solution of the homogeneous equation $y'' - 4y = 0$ is known:

$$y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$$

So the general solution of $y'' - 4y = \sin(x)$ is

$$y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{5} \sin(x)$$

Solve initial value problem

$$0 = y(0) = c_1 e^0 + c_2 e^0 - \frac{1}{5} \sin(0) = c_1 + c_2$$

$$y'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x} - \frac{1}{5} \cos(x)$$

$$0 = y'(0) = 2c_1 e^0 - 2c_2 e^0 - \frac{1}{5} \cos(0) = 2c_1 - 2c_2 - \frac{1}{5}$$

$$\begin{aligned} c_1 + c_2 &= 0 & \Rightarrow & c_2 = -c_1 \\ 2c_1 - 2c_2 &= \frac{1}{5} & \longrightarrow & 4c_1 = \frac{1}{5} \quad c_1 = \frac{1}{20} \quad c_2 = -\frac{1}{20} \end{aligned}$$

$$y(x) = \frac{1}{20} e^{2x} - \frac{1}{20} e^{-2x} - \frac{1}{5} \sin(x)$$