Linear independence and general solutions

We return to the question, when have we found the complete general solution to a linear differential equation?

2nd order Homogeneous case $\quad y'' + p(x)y' + q(x)y = 0$

We must find 2 solutions, and the must be really different from each other. The precise term is linearly independent.

Definition Two functions $y_1(x)$ and $y_2(x)$ are linearly independent if they are NOT PROPORTIONAL

$$\left.\begin{array}{l} y_1(x) \neq C y_2(x) \\ y_2(x) \neq C y_1(x) \end{array}\right\} \text{ for any constant } C.$$

Examples
- $\sin(x)$, $\cos(x)$
- e^x, e^{-2x}
- e^x, xe^x

Nonexamples
- e^x, $2e^x$
- 0, $\sin(x)$

The complete general solution of a second order linear homogeneous differential equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1 and y_2 are two linearly independent solutions