Example Find $\int \frac{4x}{x^2+4}\,dx$

Note that this is "almost" a logarithmic derivative (recall $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$). So with this inspiration use $u = x^2+4$, then $du = 2x\,dx$ or $\frac{1}{2}du = x\,dx$

$$\begin{aligned}
\int \frac{4x}{x^2+4}\,dx &= 4\int \frac{x}{x^2+4}\,dx \\
&= 4\int \frac{1}{u}\cdot\frac{1}{2}\,du \\
&= \frac{4}{2}\int \frac{1}{u}\,du \\
&= 2\ln|u| + C(\text{ go back to your original variable }) \\
&= 2\ln|x^2+4| + C
\end{aligned}$$

The Substitution Rule for Definite Integrals If $g'(x)$ is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example Calculate $\int_0^4 \sqrt{2x+1}\,dx$

As before find your u first. Here $u = 2x+1$ then $du = 2dx$ or $\frac{1}{2}du = dx$.Since we have a definite integral here the last theorem says that after the substitution we also have to find out to what u values $x=0$ and $x=4$ changed to. This is done by using your substitution: $x = 0 \Rightarrow u = 2\cdot 0+1 = 1$
$x = 4 \Rightarrow u = 2\cdot 4+1 = 9$
So we have

$$\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{2}\int_1^9 \sqrt{u}\,du$$