Section 4.6 Integration by Substitution

In this section we will be reversing the "chain rule". Recall that the Chain Rule says

$$\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x)$$

So the general anti derivative of $f'(g(x)) \cdot g'(x)$ is $f(g(x)) + C$ or

$$\boxed{\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C}$$

Example Calculate $\int 2x\sqrt{1+x^2}\, dx$

Comparing with the formula we derived above here $f'(x) = \sqrt{x}$ and $g(x) = 1 + x^2$ (note $g'(x) = 2x$). Since anti derivative of $\sqrt{x}$ is $\frac{x^{3/2}}{3/2} + C$, so $f(x) = \frac{2}{3}x^{3/2} + C$. Then by the formula we have

$$\int 2x\sqrt{1+x^2}\, dx = f(g(x)) + C = \frac{2}{3}(1+x^2)^{3/2} + C$$

The process above is some-what notation wise messy. There is a systematized, less "notation" way of doing this which is commonly referred to as u-substitution. We will re-do the first example using the "u-substitution" to show you how it works.

Example 1-Revisited Like in the chain rule, we will recognize $1+x^2$ as the inner-function and we will define $u = 1 + x^2$. Then recall the definition of differentials from Section 3.1 If $y = f(x)$, the differential of y is $dy = f'(x)dx$ We will use this recall here for $u = 1+x^2$ and get $du = 2x\, dx$. (A side remark here: Until now, we have only thought of the "dx" in the integrand as a place holder. Now, think of it as a differential.)

$$\begin{aligned}
\int 2x\sqrt{1+x^2}\, dx &= \int \sqrt{1+x^2}(2x\, dx) \\
&= \int \sqrt{u}\, du \\
&= \frac{2}{3}u^{3/2} + C \text{ (go back to your original variable)} \\
&= \frac{2}{3}(1+x^2)^{3/2} + C
\end{aligned}$$