Section 4.5 Fundamental Theorem of Calculus

Definition:(A function defined by Integration) Suppose f is continuous on an interval containing "a", we define
$\Phi(x) = \int_a^x f(s)\, ds$

<u>Remarks:</u> 1) $\Phi(x)$ is continuous (small changes in x ????)
2)If $f(x) > 0$ then $\Phi(x)$ is increasing, if $f(x) < 0$ then $\Phi(x)$ is decreasing.
3) $\Phi(a) = 0$ and $\Phi(b) = \int_a^b f(s)\, ds$

Examples 1)If $f(x) = \frac{1}{x}$ and $a = 1$ then $\Phi(x) = \int_1^x \frac{1}{s}\, ds$

2)If $f(x) = x$ and $a = 0$ then $\Phi(x) = \int_0^x t\, dt = \frac{1}{2}x^2$

Theorem(1st Version of Fundamental Theorem of Calculus)
Let f be continuous on an interval containing "a", then for any x in this interval

$$\boxed{\frac{d}{dx}\Phi(x) = \frac{d}{dx}\int_a^x f(s)\, ds = f(x)}$$

Proof. Let's take the derivative of Φ using the definition of the derivative:

$$\begin{aligned}
\Phi'(x) &= \lim_{h\to 0} \frac{\Phi(x+h) - \Phi(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(s)\, ds - \int_a^x f(s)\, ds\right) \\
&= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(s)\, ds \\
&= \lim_{h\to 0} \frac{1}{h} f(c)\cdot h \text{ for some } x < c < x+h \text{ by the MVT for Integrals} \\
&= \lim_{h\to 0} f(c) = f(x) \text{ since f is continuous.}
\end{aligned}$$

Examples 1) $\frac{d}{dx}\int_1^x \frac{1}{s}\, ds = \frac{1}{x}$
2) $\frac{d}{dx}\int_1^x \sqrt{1+s^4}\, ds = \sqrt{1+x^4}$
3) $\frac{d}{dt}\int_1^t \sin(x^2)\, dx = \sin(t^2)$
4) $\frac{d}{dr}\int_1^r \frac{t^2}{\sqrt{1+t^2}}\, ds = \frac{t^2}{\sqrt{1+r^2}}$