Choose $c_i^\star = \frac{c_{i-1}+c_i}{2}$ Mid-point of each subinterval. Note here we will replace $\Delta x = c_i - c_{i-1} = \frac{b-a}{n}$ which is length of each subinterval to simplify our calculations.

$$\begin{aligned}\sum_{i=1}^{n} f\left(\frac{c_{i-1}+c_i}{2}\right)\Delta x &= \sum_{i=1}^{n} \frac{c_{i-1}+c_i}{2}\cdot(c_i - c_{i-1})\\ &= \sum_{i=1}^{n} \frac{c_i^2 - c_{i-1}^2}{2}\\ &= \frac{1}{2}\Big((x_1^2 - x_0^2) + (x_2^2 - x_1^2) + (x_3^2 - x_2^2) + \dots + (x_n^2 - x_{n-1}^2)\Big)\\ &= \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}(b^2 - a^2) \text{ because } x_n = b, x_0 = a\end{aligned}$$

Take the limit $n \to \infty$

$$\begin{aligned}\int_a^b x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{c_{i-1}+c_i}{2}\right)\Delta x\\ &= \lim_{n\to\infty} \frac{1}{2}(b^2 - a^2) = \frac{b^2 - a^2}{2}\end{aligned}$$

Choose $c_i^\star = c_i$ Right-hand endpoint of each subinterval.

$$\begin{aligned}\sum_{i=1}^{n} f(c_i^\star)\Delta x &= \sum_{i=1}^{n} c_i \cdot \frac{b-a}{n}\\ &= \sum_{i=1}^{n}\left(a + i\frac{b-a}{n}\right)\cdot\frac{b-a}{n}\\ &= a(b-a) + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2}\end{aligned}$$

Take the limit $n \to \infty$

$$\begin{aligned}\int_a^b x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i^\star)\Delta x\\ &= \lim_{n\to\infty}\left(a(b-a) + \frac{(b-a)^2}{2}\frac{n+1}{n}\right)\\ &= \frac{b^2 - a^2}{2}\end{aligned}$$

So we have showed in three different ways $\boxed{\int_a^b x\,dx = \frac{b^2 - a^2}{2}}$