Average Value of a Function and Mean Value Theorem for Integrals

Recall : Given a set of values $y_1, y_2, ..., y_n$ the average of these values is defined as $y_{avg} = \frac{y_1+y_2+...+y_n}{n}$.

Consider a Riemann Sum for a definite integral

$$\sum_{i=1}^{n} f(i) \cdot \frac{b-a}{n} = (b-a) \sum_{i=1}^{n} \frac{f(i)}{n}$$

Note that the fraction $\frac{f(i)}{n}$ in the sum on the right hand side is the Average of the function f at n-test points.This observation leads us to a more interesting one below

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(i) \cdot \frac{b-a}{n} = (b-a) \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(i)}{n}$$

Divide this equation by $(b-a)$

$$\frac{1}{b-a} \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(i)}{n}$$

So the right hand side expression in the limit gives us the Average value of f on $[a, b]$. So based on this argument

Definition If f is integrable on the interval $[a, b]$, then the average value of f on $[a, b]$ is

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Mean Value Theorem for Integrals If f is continuous on the interval $[a, b]$, then there is a c in (a, b) such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$