Section 4.4 The Definite Integral

Definition For any function f defined on the interval $[a, b]$, the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i^\star)\Delta x$$

whenever the limit exists and is the same for any choice of test points $c_i^\star$ in the i^{th} subinterval. When the limit exists, we say that f is integrable on $[a, b]$.

Remarks 1) We have defined the sum on the right hand side as the Riemann Sum in Section 4.3 and mentioned that the sum aprroaches the area under f as we let $n \to \infty$. So the definite integral we have defined above gives us the area under the curve $y = f(x)$ above the x-axis when x changes from a to b.(Note that we are assuming $f(x) \geq 0$. If the area has negative sign this means the area we are interested is below the x-axis.–More on this later)

2) Since the definite integral is ultimately a limit if we know that $\int_a^b f(x)\,dx$ exists, then to evaluate the integral we can do so by choosing a special Δx and special $c_i^\star$.

First we will evaluate couple of integrals by using the first remark. Then we will see how we calculate an integral by using limit of a Riemann sum calculation.

Example Evaluate the integrals below by interpreting each in terms of areas.

a) Calculate $\int_0^1 \sqrt{1 - x^2}\,dx$

Notice first that

$$y = \sqrt{1 - x^2} \Rightarrow y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1$$

gives us a circle centered at the origin with radius one. But the area we are interested in is only 1/4th of this circle because we want the part when x is changing from 0 to 1 as in the figure below.