Proof I will prove the rule in the case of $\frac{0}{0}$ indeterminate form. So this indeterminate form implies $f(c) = g(c) = 0$

$$\begin{aligned}
\lim_{x\to c}\frac{f(x)}{g(x)} &= \lim_{x\to c}\frac{\frac{f(x)}{x-c}}{\frac{g(x)}{x-c}} \\
&= \lim_{x\to c}\frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} \\
&= \frac{\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}}{\lim\limits_{x\to c}\frac{g(x)-g(c)}{x-c}} \\
&= \frac{f'(c)}{g'(c)} \text{ assuming } g'(c) \neq 0 \\
&= \lim_{x\to c}\frac{f'(x)}{g'(x)}
\end{aligned}$$

Example Find $\lim\limits_{x\to 1}\frac{x^{15}-1}{x^3-1}$

$\lim\limits_{x\to 1}\frac{x^{15}-1}{x^3-1} \overbrace{=}^{(\frac{0}{0})L'Hop} \lim\limits_{x\to 1}\frac{15x^{14}}{3x^2} = \frac{15}{3} = 5$

Let's also compare this with the answer we would get if we used the linear approximations we have seen in Section 3.1 instead of the L'Hop.

$x^{15} - 1 \approx 15(x-1)$ (Here $f(x) = x^{15} - 1,\ a = 1,\ f(a) = b = 0, m = f'(1) = 15$ and $f(x) \approx m(x-a) + b$).

Similarly $x^3 - 1 \approx 3(x-1)$. Therefore,

$\frac{x^{15}-1}{x^3-1} \approx \frac{15(x-1)}{3(x-1)} = 5$

Example $\lim\limits_{x\to\infty}\frac{\ln x}{x-1} \overbrace{=}^{(\frac{\infty}{\infty})L'Hop} \lim\limits_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim\limits_{x\to\infty}\frac{1}{x} = 0$