Lemma Suppose $f'(x) = 0$ for all x in some open interval I. Then $f(x)$ is constant on I.

    Proof Choose any two number a and b in the interval I such that $a < b$. We know f is differentiable on I (we even know it's derivative is zero there) so it is continuous. So we may apply MVT and we will get

$$\frac{f(b) - f(a)}{b - a} = f'(c) \text{ for some c in} (a, b)$$

But $f'(c) = 0$ always so;

$$\frac{f(b) - f(a)}{b - a} = 0 \Rightarrow f(b) - f(a) = 0 \Rightarrow f(a) = f(b)$$

for any choice of a and b (after all we didn't specify anything about them). So f is a constant function on I.