Uniqueness of the root: Suppose $f(x)$ has two (or more) distinct roots in the interval $(0, 1)$. According to the statement of the problem this assumption is wrong. So by assuming this if I run into a wall (get into a mathematical trouble I cannot get out), then problems statement is correct. Let's call these two roots as "A" and "B". So $f(A) = f(B) = 0$. We will form a new interval using A and B $[A, B]$ and apply the Rolle's Theorem on this interval. (We may apply it because f is a polynomial hence it is continuous and differentiable everywhere). By Rolle's theorem there is k in the interval (A, B) such that
$f'(k) = 0 \Rightarrow 6k^2 + 3 = 0$

Note that this equation has no solution, so there is no such k. But Rolle's Theorem promised one. So I hit that "mathematical trouble/wall". I have to go back and accept the defeat and say that there is only one root in the interval $[0, 1]$ as two gave me headache.

Example Suppose $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

Assuming f is continuous and differentiable apply MVT on some interval that contains 0 and 2. So we get

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) \text{ for some c}$$

$$f(2) - f(0) = 2f'(c)$$

$$f(2) = 2f'(c) + f(0)$$

Now use the information given on the derivative $f'(x) \leq 5$, so $f'(c) \leq 5$.

$$f(2) \leq 2 \cdot 5 + (-3) = 7$$

The largest possible value for $f(2)$ is 7.

Apart from applications such as the ones above, MVT is a strong tool. It will to help us prove some facts which will help us to use the functions derivative itself to make conclusion about the actual function. Below you will see such a fact