Applications of Mean Value Theorem

Example Consider the function $f(x) = x^3 - x$ on $[0, 2]$. Show that it satisfies MVT first and secure the existence of c. Then find c (if possible).

$f(x)$ is a polynomial which is continuous and differentiable every where hence in particular on our interval. So by MVT there is a "c" in the interval $(0, 2)$ for which

$$\frac{f(2) - f(0)}{2 - 0} = \frac{(2^2 - 2) - 0}{2 - 0} = 3 = f'(c)$$

Since $f'(x) = 3x^2 - 1$ we will be able to find c exactly in this case:

$$f'(c) = 3 \Rightarrow 3c^2 - 1 = 3 \Rightarrow 3c^2 = 4 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm\frac{2}{\sqrt{3}}$$

Be careful here though, only $\frac{2}{\sqrt{3}}$ is the only one of these that is in the interval $(0, 2)$. So $c = \frac{2}{\sqrt{3}}$

Example Prove that $f(x) = 2x^3 + 3x - 1$ has exactly on root in the interval $[0, 1]$.

First let's understand the problem well. The questions wants us to prove there is a root in this interval and the root is unique. It does NOT expect us to find the root. That is a task we will leave for another Chapter.
Showing existence and then proving uniqueness will require a united power of two important theorems. We will show existence by using Intermediate Value Theorem and the we will prove the uniqueness of this root by Rolle's Theorem.

Existence of the root: Note that $f(x)$ is a polynomial and $f(1) > 0$ and $f(0) < 0$, so by Intermediate Theorem there is a root of the polynomial $f(x)$ in the interval $(0, 1)$.

Now we know there is at least one root in this interval. But question says there is exactly one. Rolle's Theorem will take care of that part for us as follows;