Example Find $\frac{dy}{dx}$ for the graph $\sqrt{1+x^2y^2} = 2xy$.

In this problem I'm going to use the shorthand $y' = \frac{dy}{dx}$ notation. It keeps the calculation "less crowded" with notations.This is a typical implicit differentiation problem. We begin by computing the derivatives of the left and right hand sides. For the right hand side, we have

$$\frac{d}{dx}[2xy] = 2y + 2xy'$$

(note: I had to use the product rule to compute this derivative). For the left hand side, I notice that $\sqrt{1+x^2y^2} = f(g(x))$ where $f(x) = \sqrt{x}$ and $g(x) = 1 + x^2y^2$. Hence the chain rule says that

$$\frac{d}{dx}[\sqrt{1+x^2y^2}] = \frac{1}{2\sqrt{1+x^2y^2}}(x^2 2yy' + 2xy^2) = \frac{2x^2yy'}{2\sqrt{1+x^2y^2}} + \frac{2xy^2}{2\sqrt{1+x^2y^2}}$$

Setting the two derivatives equal to each other, I solve for y'

$$\begin{aligned}
2y + 2xy' &= \frac{2x^2yy'}{2\sqrt{1+x^2y^2}} + \frac{2xy^2}{2\sqrt{1+x^2y^2}} \Rightarrow \\
y - \frac{xy^2}{2\sqrt{1+x^2y^2}} &= \frac{x^2yy'}{2\sqrt{1+x^2y^2}} - xy' \Rightarrow \\
y - \frac{xy^2}{2\sqrt{1+x^2y^2}} &= y'\left(\frac{x^2y}{2\sqrt{1+x^2y^2}} - x\right) \Rightarrow \\
y' &= \frac{y - \frac{xy^2}{2\sqrt{1+x^2y^2}}}{\frac{x^2y}{2\sqrt{1+x^2y^2}} - x}
\end{aligned}$$