Implicit Differentiation

To solve these problems, we develop a new way to evaluate derivatives of implicit functions of x.

Example Find the slope of the line tangent to the graph $x^2 + y^2 = 1$ at an arbitrary point (x, y) on the curve.

We're going to use a new tool called implicit differentiation to solve this problem. Our idea will be to take the given expression $x^2 + y^2 = 1$ and evaluate the derivative of both the left and right hand sides. In doing this, the desired derivative (which is $\frac{dy}{dx}$, since this represents rise over run) will pop out, and well be able to solve.

The derivative of the right hand side of our expression is easy: $\frac{d}{dx}[1] = 0$. The left hand side is slightly more complicated: $\frac{d}{dx}[x^2 + y^2] = 2x + 2y\frac{dy}{dx}$. Why does this strange factor of $\frac{dy}{dx}$ show up? Essentially, this is just the chain rule. In this case the variable y is implicitly a function of x, and so when we evaluate its derivative we need to use the chain rule. To see how it appears, lets write $y = f(x)$. Then

$$\frac{d}{dx}[y^2] = \frac{d}{dx}[(f(x))^2] = 2f'(x)f(x) = 2y\frac{dy}{dx}$$

Hence we have the stated equality: $\frac{d}{dx}[x^2 + y^2] = 2x + 2y\frac{dy}{dx}$.

Setting the derivatives of the left and right hand sides equal gives $2x + 2y\frac{dy}{dx} = 0$. Now since were after $\frac{dy}{dx}$ we can just solve:

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

This means that the slope of the line tangent to the graph $x^2 + y^2 = 1$ at a point (x, y) is given by $\frac{-x}{y}$. As a reality check, we can verify this formula against what we know the slopes of the tangent to the circle at certain nice points to be.