Section 2.8 Implicit Differentiation

One of the most basic objects in mathematics is the circle. Analytically, a circle of radius 1 centered at the origin is represented by the formula $x^2 + y^2 = 1$. As it is one of the fundamental mathematical graphs, it is natural to want to solve the tangent problem for the circle. Sadly, however, the graph of the circle is not the graph of a function because it fails the vertical line test. Since we only have tools for solving the tangent problem for functions, we can't yet solve the tangent problem for the circle.

How can a person remedy this problem? One trick would be to take the expression for the circle and solve for y. This would give an expression for y in terms of x, an expression which we might then be able to evaluate the derivative of. In this case, solving for y gives $y = \pm\sqrt{1 - x^2}$. The + and - are there because we've split the circle into a top piece $y = \sqrt{1 - x^2}$ and a bottom piece $y = -\sqrt{1 - x^2}$. We could then evaluate the derivative of each of these functions and then use them to find the slope of the line tangent to a point (x, y) on the circle (which derivative we used would depend on which half of the circle (x, y) lived on). This would result in derivatives

$$-\frac{x}{\sqrt{1 - x^2}} \text{ and } \frac{x}{\sqrt{1 - x^2}}$$

There are a few problems with this approach

• From an aesthetic standpoint, its pretty flunky and unnatural. Given an expression thats nice like $x^2 + y^2 = 1$, it's silly that we need to evaluate ugly derivatives like $\sqrt{1 - x^2}$

• From a practical standpoint, it's problematic because to find the slope of the tangent to the curve at a point P we have to figure out which of the two functions P "lives on". This is not too much of a problem for the circle, but can become more complicated when we have nastier expressions.

• From a computability standpoint, given a complicated expression involving x's and y's, there is no guarantee that we will even be able to solve for y! Nevertheless, the tangent problem still makes perfect sense for these complicated graphs, so we need a new tool.