$$\frac{1}{\cos(\theta)} > \frac{\theta}{\sin(\theta)} > 1$$

Take reciprocals:

$$\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$$

Now since $\lim_{\theta\to 0} \cos(\theta) = 1$ and $\lim_{\theta\to 0} 1 = 1$ by Squeeze Theorem

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$$

Now if you go back to the derivative of $\sin(x)$ at $x = 0$;

$$\frac{d(\sin(x))}{dx}|_{x=0} = \sin'(0) = \lim_{h\to 0} \frac{\sin(h)}{h} = 1$$

Lemma $\lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta} = 0$

Proof Note first that the first attempt to plug in $\theta = 0$ gives you the indeterminate form $\frac{0}{0}$. To evaluate this limit we will use the result we got in the above lemma as well as the trig property $\cos^2(\theta) + \sin^2(\theta) = 1 \Rightarrow \sin^2(\theta) = \cos^2(\theta) - 1$.

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta} &= \lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta} \cdot \frac{\cos(\theta)+1}{\cos(\theta)+1} \\
&= \lim_{\theta\to 0} \frac{\cos^2(\theta)-1}{\theta(\cos(\theta)+1)} \\
&= \lim_{\theta\to 0} \frac{\sin^2(\theta)}{\theta(\cos(\theta)+1)} \\
&= \lim_{\theta\to 0} (\frac{\sin(\theta)}{\theta} \cdot \frac{\sin(\theta)}{\cos(\theta)+1}) \\
&= \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta\to 0} \frac{\sin(\theta)}{\cos(\theta)+1} \\
&= 1 \cdot 0 = 0
\end{aligned}$$

Now let's first try to find the derivative of $\cos(x)$ at $x = 0$;

$$\cos'(0) = \lim_{h\to 0} \frac{\cos(0+h) - \cos(0)}{h} = \lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$$