How to correct the flaw in the Proof of Chain Rule

We will start by asking: So what if g is a function for which $g(x) = g(a)$ for lots of numbers $x \neq a$? The Chain Rule should hold in this case as well. How can we fix the flaw and make a proof that works for all differentiable functions g?

One way to proceed is to make a new function that is like: $\frac{f(g(x))-f(g(a))}{g(x)-g(a)}$ (the source of the problem), but which won't care whether g(x) equals g(a) or not. This can be done as follows. Start with [1]

$$F(u) = \begin{cases} \frac{f(u)-f(g(a))}{u-g(a)} & \text{if } u \neq g(a) \\ f'(g(a)) & \text{if } u = g(a) \end{cases} \tag{1}$$

So that for any x,

$$F(g(x)) = \begin{cases} \frac{f(g(x))-f(g(a))}{g(x)-g(a)} & \text{if } g(x) \neq g(a) \\ f'(g(a)) & \text{if } g(x) = g(a) \end{cases} \tag{2}$$

Clearly the definition of $f'(g(a))$ and (1) guarantee that $\lim_{u \to g(a)} F(u) = F(g(a))$ (i.e. F is continuous at "g(a)"); therefore, since $\lim_{x \to a} g(x) = g(a)$, the Continuity Theorem (2nd Theorem in the lectures of Section 1.4) implies that

$$\lim_{x \to a} F(g(x))) = F(\lim_{x \to a} g(x)) = F(g(a)) \overbrace{=}^{by\ (2)} f'(g(a)) \tag{3}$$

Moreover, (2) implies the following lemma, which says that F(g(x)) and $\frac{f(g(x))-f(g(a))}{g(x)-g(a)}$ are similar enough to afford a route around the flaw in the crucial limit computation.

Lemma For any $x \neq a$

$$\underbrace{\frac{f(g(x)) - f(g(a))}{x - a}}_{(E)} = F(g(x)) \cdot \underbrace{\frac{g(x) - g(a)}{x - a}}_{(D)} \tag{4}$$

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[1] Here, "u" is a pure variable rather than a stand-in for g(x).