Example Let $g(t) = \left(\frac{t-2}{2t+1}\right)^9$. Find $g'(t)$.

$y = g(t)$ decomposes into $y = u^9$ and $u(t) = \frac{t-2}{2t+1}$

$$\frac{dg}{dt} = \frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \underbrace{\frac{d}{dt}(\frac{t-2}{2t+1})}_{\text{needs Quotient Rule}}$$

$\frac{d}{dt}(\frac{t-2}{2t+1}) = \frac{d}{dt}(\frac{f}{g}) = \frac{f'g-fg'}{[g]^2}$; since $f'(t) = 1$ and $g'(t) = 2 \Rightarrow$
$\frac{d}{dt}(\frac{t-2}{2t+1}) = \frac{1\cdot(2t+1)-(t-2)\cdot 2}{(2t+1)^2}$

$$\frac{dg}{dt} = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \left[\frac{1\cdot(2t+1)-(t-2)\cdot 2}{(2t+1)^2}\right] = \frac{45(t-2)^8}{(2t+1)^{10}}$$

Example Let $y = (2x+1)^5(x^3-x+1)$ and find y'.

$$y' = \underbrace{\frac{d}{dx}[(2x+1)^5]}_{\text{Chain Rule}} \cdot (x^3-x+1) + (2x+1)^5 \cdot \frac{d}{dx}[(x^3-x+1)] \text{ by Product Rule}$$

$$= \overbrace{5(2x+1)^4 \cdot (2x)} \cdot (x^3-x+1) + (2x+1)^5(3x^2-1)$$

Theorem(Derivative of inverse functions) If f(x) is differentiable at all x, and has an inverse $g(x) = f^{-1}(x)$, then

$$\boxed{g'(x) = \frac{1}{f'(g(x))} \text{ provided } f'(g(x)) \neq 0}$$

Proof Recall that if f and g are inverses, then $f(g(x)) = x$ for all x in the domain of g. Differentiate both sides of this equality with respect to x:

$$\underbrace{\underbrace{\frac{d}{dx}[f(g(x))]}_{\text{by Chain Rule}}}_{f'(g(x))\cdot g'(x) = 1} = \frac{d}{dx}[x]$$

$$\text{Hence } g'(x) = \frac{1}{f'(g(x))}$$