Theorem (The Chain Rule) If g is differentiable at x and f is differentiable at g(x), then

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)}$$

Sometimes it is easier to remember the Chain Rule in Leibnitz notation. If $y = f(g(x))$ then $\frac{dy}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Flawed Proof We must evaluate

$$[f(g(a))]' = (f \circ g)'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

in order to do this we need to find a way to separate the contribution of f from the contribution of g to this limit. The way I will do this is to multiply and divide by the quantity $[g(x) - g(a)]$:

$$\begin{aligned}\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} &= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \\ &= \underbrace{\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}}_{(A)} \cdot \underbrace{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}}_{(B)}\end{aligned}$$

Now I will evaluate limits (A) and (B) separately. Limit (B) is obviously $g'(a)$; nothing more needs to be said about that. As for limit (A), notice that as $x \to a$, $g(x) \to g(a)$, because g is continuous at a (it is differentiable remember!!).This means that

$$(A) = \lim_{x \to a_{g(x) \to g(a)}} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = \underbrace{\lim_{g(x) = u \to g(a)} \frac{f(u) - f(g(a))}{u - g(a)}}_{(C)}$$

(In (C) I let u stand for g(x).) Now as $u = g(x) \to g(a)$, by the definition of the derivative, (C) must be approaching $f'(g(a))$. Thus the limit (A) equals $f'(g(a))$ and the product of the two limits (A) and (B) is: $f'(g(a)) \cdot g'(a)$