Section 2.5 Chain Rule

First we will start with couple of motivational examples:

Example Find $\frac{d}{dx}[(x^2+1)^2] = \frac{d}{dx}[(x^2+1)(x^2+1)]$.
The second part of the problem is suggesting the use of the Product Rule.

$$\begin{aligned}\frac{d}{dx}[(x^2+1)^2] &= \frac{d}{dx}[(x^2+1)(x^2+1)]\\ &= (2x)(x^2+1)+(x^2+1)(2x)\\ &= 2\cdot(x^2+1)\cdot 2x\end{aligned}$$

Example Find $\frac{d}{dx}[(x^2+1)^3] = \frac{d}{dx}[(x^2+1)(x^2+1)^2]$.
Another call for Product Rule. We will also need the help of the previous example:

$$\begin{aligned}\frac{d}{dx}[(x^2+1)^3] &= \frac{d}{dx}[(x^2+1)(x^2+1)^2]\\ &= 2x(x^2+1)^2+(x^2+1)[2\cdot(x^2+1)\cdot 2x]\\ &= 3\cdot(x^2+1)^2\cdot 2x\end{aligned}$$

By similar argument as above,
$\frac{d}{dx}[(x^2+1)^4] = 4\cdot(x^2+1)^3\cdot 2x$
$\frac{d}{dx}[(x^2+1)^5] = 5\cdot(x^2+1)^4\cdot 2x$
$\frac{d}{dx}[(x^2+1)^6] = 6\cdot(x^2+1)^5\cdot 2x$

Notice a pattern here? $\frac{d}{dx}[(x^2+1)^n] = n\cdot(x^2+1)^{n-1}\cdot 2x$.
Here our original function $(x^2+1)^n = f\circ g(x)$ is composition of two functions where $f(u)=u^n$ is the "outer" function and $u(x)=x^2+1$ is the inner function. So we can re-interpret the pattern above as:
$$\frac{d}{dx}[(x^2+1)^n] = \underbrace{n\cdot(x^2+1)^{n-1}}_{\text{derivative of outer function at } u(x)} \cdot \underbrace{2x}_{\text{derivative of inner function}}$$