Section 2.4 Product and Quotient Rules

Product Rule

Last class period we saw that the derivative of a sum of functions is the sum of the derivative of each function. Is the same true for products? Lets investigate $\frac{d}{dx}[f \cdot g]$ with an example. Lets take $f(x) = g(x) = x$. Then we have

$$\frac{d}{dx}[f \cdot g] = \frac{d}{dx}[x^2] = 2x^{2-1} = 2x$$

and

$$\frac{d}{dx}[f]\frac{d}{dx}[g] = \frac{d}{dx}[x]\frac{d}{dx}[x] = 1 \cdot 1 = 1$$

In this case we have $\frac{d}{dx}[f \cdot g] \neq \frac{d}{dx}[f]\frac{d}{dx}[g]$.So what is the derivative of a product of functions?

Theorem (The Product Rule) Suppose f and g are differentiable functions. Then

$$\boxed{\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)}$$

Proof

$$\begin{aligned}
\frac{d}{dx}[f \cdot g] &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x+h) - \overbrace{f(x)g(x+h) + f(x)g(x+h)}^{\text{adding 0 to expression}} - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h} \\
&= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right) \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} g(x+h) + \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}
\end{aligned}$$

(Since each individual limit exists, we may use Limit Laws 1 and 3 above)

$$= \underbrace{\frac{d}{dx}[f(x)]}_{\text{by defn. of deriv.}} \cdot \underbrace{g(x)}_{\text{g diff. so cont}} + \underbrace{f(x)}_{\text{f diff. so cont.}} \cdot \underbrace{\frac{d}{dx}[g(x)]}_{\text{by defn. of deriv.}}$$