Example The position of a particle is given by the equation of motion $s(t) = \frac{1}{t+1}$ where t is measured in seconds and s in meters.
a) Find the velocity function $s'(x)$
b) Find the velocity of the particle at $t = 2, 5$ and $t = 10$ seconds.

a)

$$\begin{aligned} s'(t) &= \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{1+(t+h)} - \frac{1}{1+t}}{h} \\ &= \lim_{h\to 0} \frac{1+t-(1+t+h)}{(1+t+h)(1+t)} \\ &= \lim_{h\to 0} \frac{-h}{h(1+t+h)(1+t)} \\ &= \lim_{h\to 0} \frac{-1}{(1+t+h)(1+t)} = \frac{-1}{(1+t)^2} \end{aligned}$$

b)

$$\begin{aligned} s'(2) &= \frac{-1}{(1+2)^2} = \frac{-1}{9} \\ s'(5) &= \frac{-1}{(1+5)^2} = \frac{-1}{36} \\ s'(10) &= \frac{-1}{(1+10)^2} = \frac{-1}{121} \end{aligned}$$

Theorem If f is differentiable at a, then f is continuous at a.

Proof To prove continuity, we need to show $\lim\limits_{x\to a} f(x) = f(a)$. This is equivalent to show that $\lim\limits_{x\to a} f(x) - f(a) = 0$. But since f(a) does not depend on x, we can push it inside the limit

$$\bullet \quad \lim_{x\to a}(f(x) - f(a)) = 0$$

So if we show • we will prove the continuity.

$$\begin{aligned} \lim_{x\to a}(f(x) - f(a)) &= \lim_{x\to a}(f(x) - f(a))\frac{x-a}{x-a} \\ &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right) (*) \end{aligned}$$