We said $\lim_{x \to x_0} f(x) = L$ means if x is near x_0 then f(x) is near L.

In a way we are playing a 2-person game between you and I, in which I claim that the $lim_{x \to x_0} f(x) = L$.You take the control of the y-axis and I (the defender) take charge of the x-axis. I ask:
Tell me what near means to you. So you will give me a constant ϵ *which will determine a distance criteria between f(x) and L and you declare* $|f(x) - L| < \epsilon$ *is near enough for you. Then it is my turn. My tactic is to choose a "nearness criterion" (*δ*) so that when x is close to* x_0 *according to my "nearness criterion", the corresponding f(x) is close to L according to your near. We play the game until one of the two things happens : either you run out of all possible your near's you can think of (in which case I was able to find my "nearness criterion" every single time and hence prove the limit is L) or for one of your near's I got stumped and was not able to find my "nearness criterion", hence you win and the function f does not have the limit L as x approaches* x_0*.*
In class I have also used the analogy of a soccer game. On the y-axis around the point L you set a net, and you get to set the "width" (ϵ) of this net. Right after you set your net, it is my turn. My job is to define a counter-width δ, that will help me to set an interval on the x-axis around x_0. For me to win the game I have to choose the width (δ) of my interval so that when I put my ball on any point in this interval and shoot it towards the function, it reflects from f towards y-axis and gets into your net and hence I score. When I am able to do this for any width of your net, then the limit exists. If I fail, even for one of your choices, then it doesn't.

Below I will give the formal definition of limit. While reading the definition keep on the side of your mind the game above:
Definition: $\lim_{x \to x_0} f(x) = L$ means for every $\epsilon > 0$ we can find a $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - L| < \epsilon$
Let's see two examples below: