Example Find $\lim\limits_{x\to 1} \arcsin(\frac{1-\sqrt{x}}{1-x})$

Since $\arcsin(\frac{1-\sqrt{x}}{1-x})$ is not defined at x=1 we can't simply push $x = 1$ into the function. But, if $\lim\limits_{x\to 1} \frac{1-\sqrt{x}}{1-x} = b$ and $\arcsin(x)$ is continuous at b, we can use the last theorem.

$$\begin{aligned}
\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} &= \lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} \cdot (\frac{1+\sqrt{x}}{1+\sqrt{x}}) \\
&= \lim_{x\to 1} \frac{1+\sqrt{x}-\sqrt{x}-x}{(1-x)(1+\sqrt{x})} \\
&= \lim_{x\to 1} \frac{1-x}{(1-x)(1+\sqrt{x})} \\
&= \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}
\end{aligned}$$

Recall that the domain of arcsin(x) is $[-1,1]$. Since 1/2 is in this domain $\arcsin(x)$ is continuous at $x = 1$. By last theorem,

$$\lim_{x\to 1} \arcsin(\frac{1-\sqrt{x}}{1-x}) = \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right) = \arcsin(\frac{1}{2}) = \frac{\pi}{6}$$

Theorem If g is continuous at a and f is continuous at g(a), then the composite function $f \circ g(x) = f(g(x))$ is continuous at a.

Example Where is $F(x) = \ln(1+\cos(x))$ continuous?

Note that $F(x) = f(g(x))$ where $f(x) = \ln(x)$ and $g(x) = 1+\cos(x)$. $F(x)$ is only defined when $1+\cos(x) > 0$. So $F(x)$ is undefined when $\cos(x) = -1$ or $x = \pm\pi, \pm 3\pi, \pm 5\pi, ...$ So F is continuous on the intervals between these values.