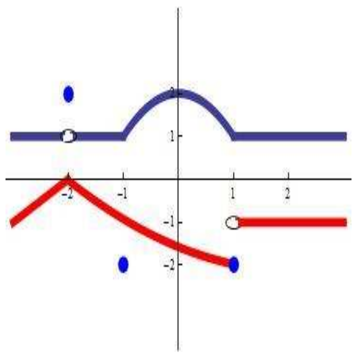
Question Does $\lim_{x\to a} f(x)g(x)$ exist even though $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not.

Answer It again depends on $f(x)$ and $g(x)$. Take your $f(x) = x$ and $g(x) = \frac{1}{x}$. $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0} 1/x$ does not exist. But $\lim_{x\to 0} x \cdot 1/x = 1$ and exists. (Note that $x \cdot 1/x$ is not defined at $x = 0$ because $1/x$ is not.)

Question What is $\lim_{x\to 1}[3f(x) + 2g(x)]$ where f(x) is the blue one and g(x) is the red.



Note again we cannot use the limit laws because $\lim_{x\to 1} g(x)$ does not exist. So instead note that the directional limits exist and hence you can use the limit laws for the directional limits and compare them for the existence of $\lim_{x\to 1}[3f(x) + 2g(x)]$.

(Recall: $\lim_{x\to 1}[3f(x)+2g(x)]$ exists iff $\lim_{x\to 1^-}[3f(x)+2g(x)]$ and $\lim_{x\to 1^+}[3f(x)+2g(x)]$ exists and they are equal.) Now:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 1, \lim_{x\to 1^-} g(x) = -2 \text{ and } \lim_{x\to 1^+} g(x) = -1$$

Using our limit laws for directional limits, we have

$$\lim_{x\to 1^-}[3f(x) + 2g(x)] = \lim_{x\to 1^-}[3f(x)] + \lim_{x\to 1^-}[2g(x)] = 3 \cdot 1 + 2 \cdot (-2) = -1$$

and

$$\lim_{x\to 1^+}[3f(x) + 2g(x)] = \lim_{x\to 1^+}[3f(x)] + \lim_{x\to 1^+}[2g(x)] = 3 \cdot 1 + 2 \cdot (-1) = 1$$

Since these two directional limit do not agree, we conclude that $\lim_{x\to 1}[3f(x) + 2g(x)]$ does not exist.

So that you have more fun while solving the hw problems and in discussion sessions I'll state the following result from your book without proof.