Coordinate Transformations and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \xrightarrow{\quad T \quad} \qquad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}, \qquad \bar{B} = TB, \qquad \bar{C} = CT^{-1}$

Claim: Controllability doesn't change.

Proof: For any $k = 0, 1, \ldots,$

$$\bar{A}^k \bar{B} = (TAT^{-1})^k TB = TA^k T^{-1} TB = TA^k B \qquad \text{(by induction)}$$

Therefore,
$$\begin{aligned} \mathcal{C}(\bar{A}, \bar{B}) &= [TB \,|\, TAB \,|\, \ldots \,|\, TA^{n-1}B] \\ &= T[B \,|\, AB \,|\, \ldots \,|\, A^{n-1}B] \\ &= T\mathcal{C}(A, B) \end{aligned}$$

Since $\det T \neq 0$, $\det \mathcal{C}(\bar{A}, \bar{B}) \neq 0$ if and only if $\det \mathcal{C}(A, B) \neq 0$.

Thus, the new system is controllable if and only if the old one is.